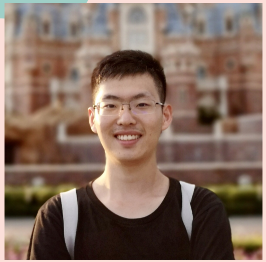
Computing a Fixed Point of Contraction Maps in Polynomial Queries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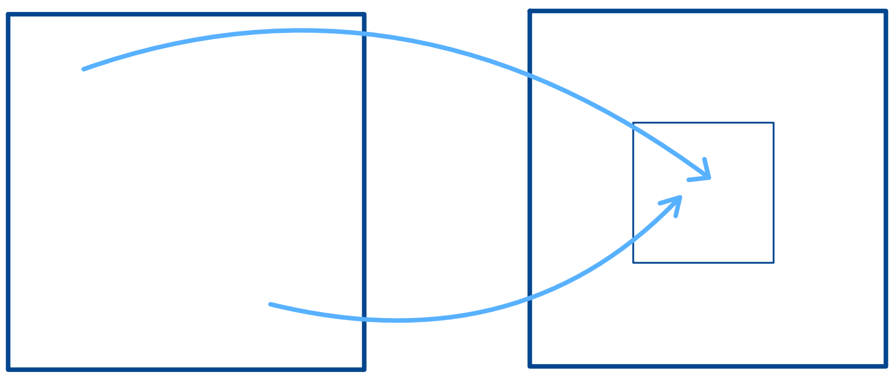
Game and Equilibria workshop, 2024

CONTRACTION FIXED POINT

$\gamma \in (0,1]$

Def. A map $f : [0,1]^k \mapsto [0,1]^k$ is a $(1-\gamma)$-contraction if

$$\|f(x) - f(y)\|_\infty \leq (1-\gamma)\,\|x-y\|_\infty \quad \forall\, x, y \in [0,1]^k.$$



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Theorem. [Banach (1922)]

Every contraction map has a unique fixed point.

$x^* = f(x^*)$

APPLICATIONS OF BANACH FIXED POINT

Mathematics:

Picard-Lindelof (Cauchy-Lipschitz) theorem

Nash embedding theorem

Computer science:

Markov decision processes

Underlie many classic dynamic programming problems

Subsume stochastic/mean-payoff/parity games

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QUERY MODEL

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Remark on approximation.

- The exact fixed point may be irrational.
- ε- approximate fixed point suffices.

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$|f(x)-x|_\infty \le \varepsilon$

SOTA: $O(\log^k(1/\varepsilon))$. [Shellman, Sikorski 03]

Goal. $\mathrm{poly}(k, \log(1/\varepsilon), \log(1/\gamma))$.

MOTIVATION

Harder

Contraction

↑

Simple Stochastic Game

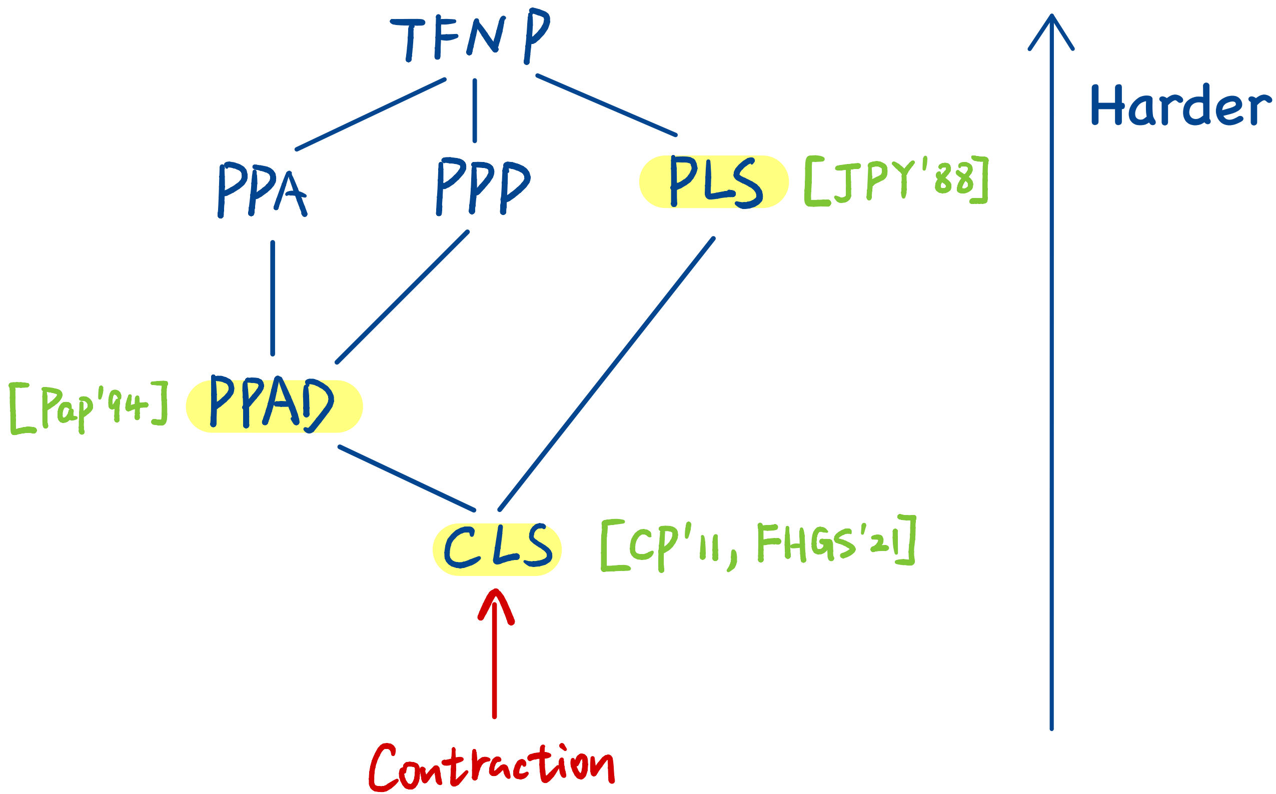
↑

Mean Payoff

↑

Parity Game

INTRIGUING STATUS



CONSTRUCTIVE EXISTENCE

Observation. Start from any point x_0 and follow the path

$$x_1 = f(x_0),\quad x_2 = f(x_1)\ \cdots$$

Then $|x_{n+1} - x_n|_\infty \le (1-\gamma)^n$.

Claim. This sequence converges to a fixed point.

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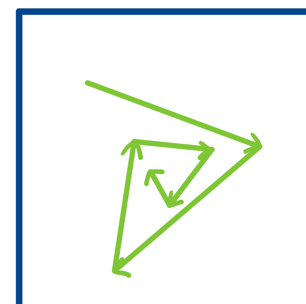
Then $|x_{n+1} - x_n|_\infty \leq (1-\gamma)^n$.

Claim. This sequence converges to a fixed point.

In fact, after $n \approx 1/\gamma \cdot \log(1/\varepsilon)$ steps, we have

$$|f(x_n) - x_n|_\infty = |x_{n+1} - x_n|_\infty \leq (1-\gamma)^n \leq \varepsilon.$$



MOTIVATION

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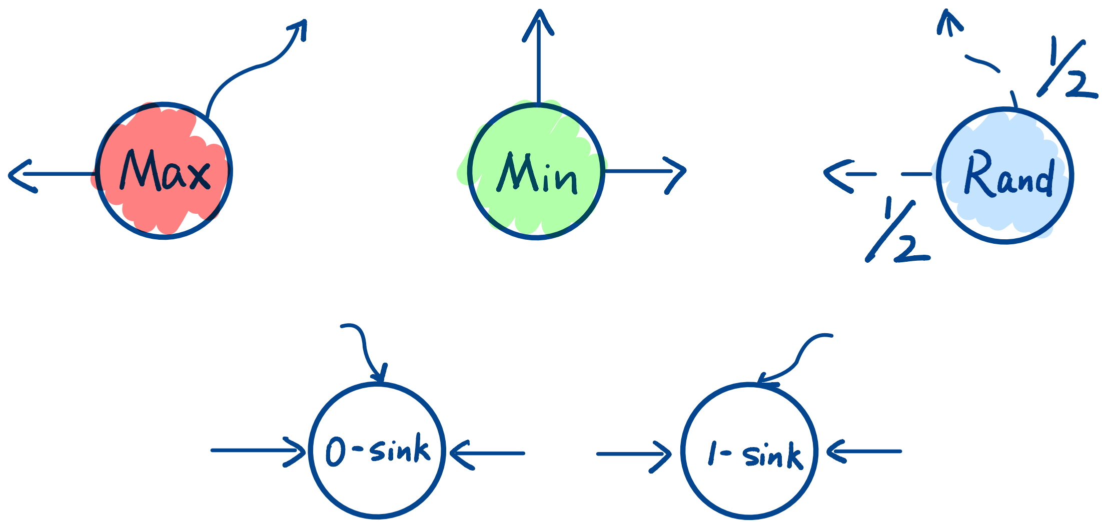
SIMPLE STOCHASTIC GAME

[Condon (1992)]

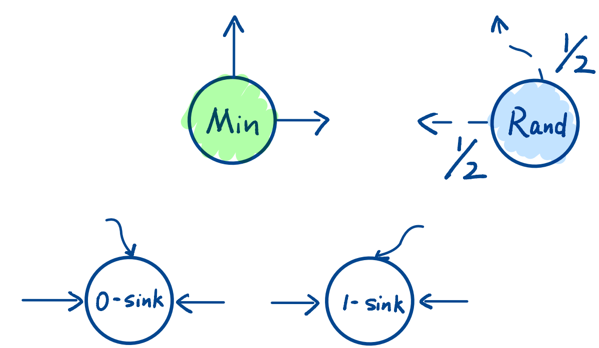


Objective: Max/Min the probability of getting to the 1-sink.

COMPLEXITY OF SSG: NP ∩ co-NP

Decision problem: if P[player 1 wins] > 1/2 .

One player version can be solved in polytime ⟹ NP ∩ co-NP.



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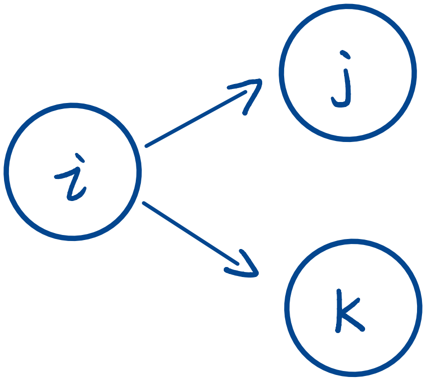
COMPLEXITY OF SSG: UP ∩ co-UP



$$v_i = \begin{cases} \max\{v_j, v_k\} & i \in V_{max} \\ \min\{v_j, v_k\} & i \in V_{min} \\ \frac{1}{2}(v_j + v_k) & i \in V_{rand} \end{cases}$$

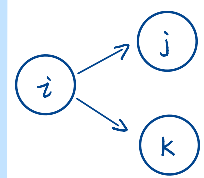
$$v_{0\text{-sink}} = 0 \qquad v_{1\text{-sink}} = 1$$

Denote this system of equations by $v = F(v)$.

COMPLEXITY OF SSG: UP ∩ co-UP

* $F : [0,1]^n \to [0,1]^n$ is a non-expansive map.

$$|F(x) - F(y)|_\infty \le |x - y|_\infty .$$



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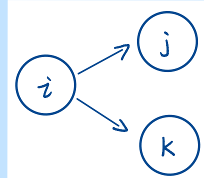

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COMPLEXITY OF SSG: UP ∩ co-UP

- Banach fixed point theorem $\Rightarrow$ unique fixed point.
- In this case, the unique fixed point is guaranteed rational + poly bit description.

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COMPLEXITY OF SSG: UP ∩ co-UP

- Banach fixed point theorem ⇒ unique fixed point.
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  rational + poly bit description.

Remark.

ε-approximate fixed point surfices.

Both ε and γ need to be $1/2^{poly(n)}$.

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MOTIVATION

Contraction $\left(\varepsilon, \gamma = 1/2^{\mathrm{poly}(n)}\right)$

↑

SIMPLE STOCHASTIC GAME

↑

MEAN PAYOFF

Pseudo-poly time

[Zwick, Paterson 96]

↑

PARITY GAME

Quasi-poly time

[Calude, Jain, Khoussainov, Li, Stephan 17]

WHY QUERY MODEL?

We have such an explicit function:

$$v_i = \begin{cases} \max\{v_j, v_k\} & i \in V_{max} \\ \min\{v_j, v_k\} & i \in V_{min} \\ \frac{1}{2}(v_j + v_k) & i \in V_{rand} \end{cases}$$

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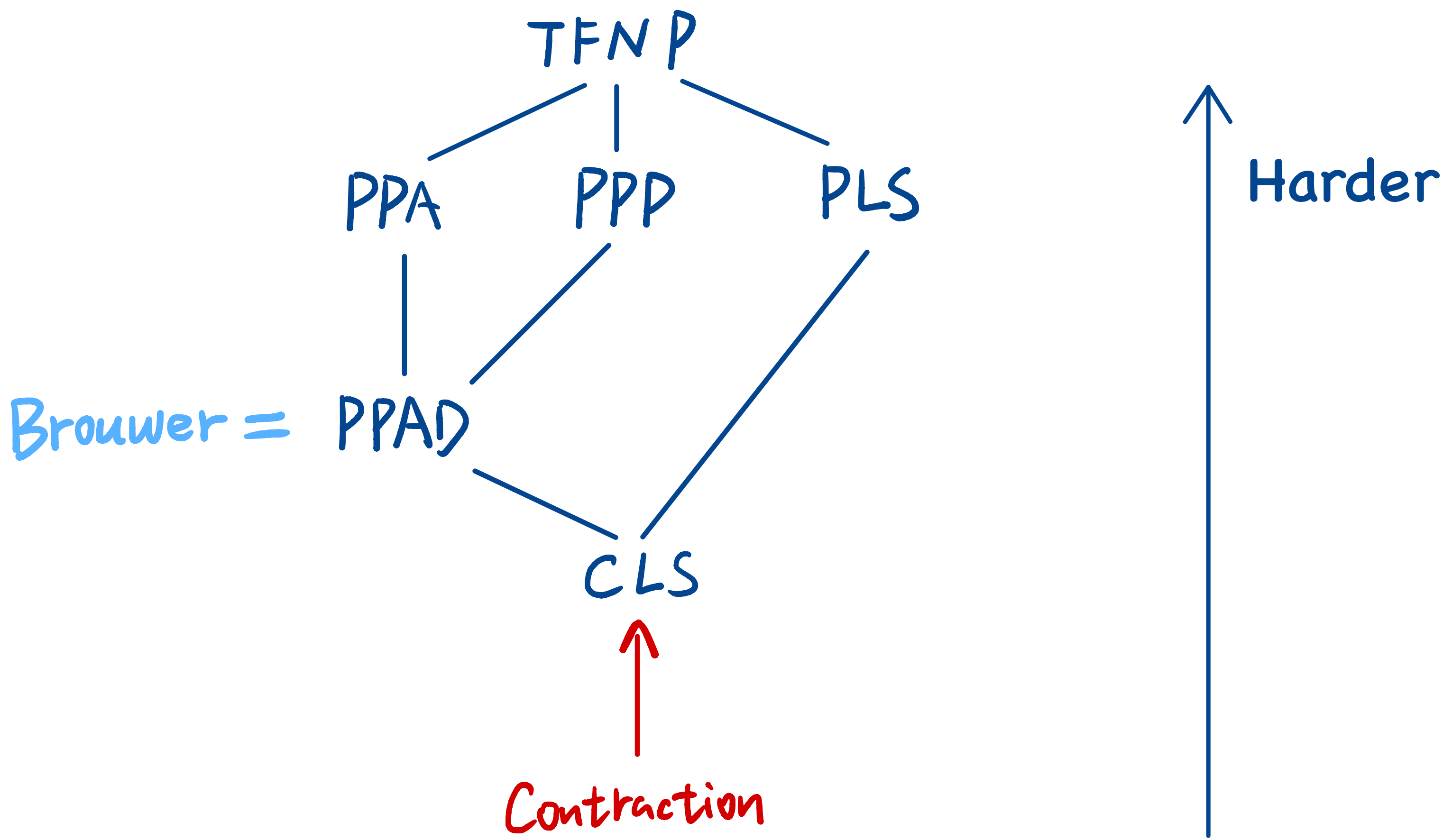
Unfortunately, we don't know how to work on them beyond evaluating function values...

Another more well-understood example: Brouwer


TFNP
PPA
PPP
PLS
Brouwer = PPAD
CLS
Contraction
Harder

BROUWER FIXED POINT

Def. A map $f:[0,1]^k \mapsto [0,1]^k$ is L-Lipschitz if

$L \in (0,\infty)$

$$|f(x)-f(y)|_\infty \le L \cdot |x-y|_\infty \quad \forall\, x,y \in [0,1]^k.$$

Theorem. [Brouwer (1911)]

Every continuous function $f: \Delta^k \to \Delta^k$ has a fixed point.

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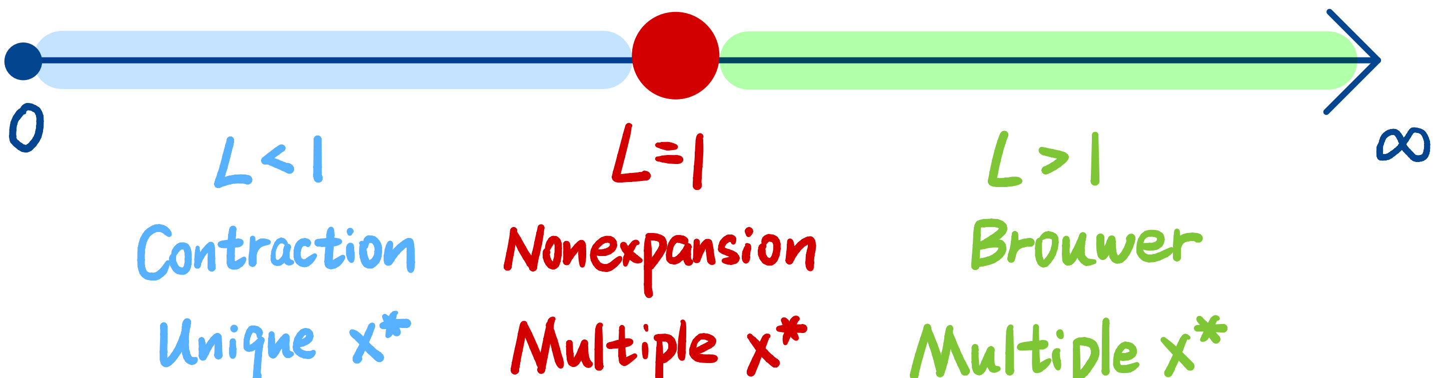

COMPLEXITY OF BROUWER

* Exponential query lower bound [HPV'89, CD'08]
* PPAD-complete (widely believed $\neq$ P)
* How about important explicit functions?

NASH EQUILIBRIUM

Theorem 23 (Nash 1951) *Every game with a finite number of players and action profiles has at least one Nash equilibrium.*

Proof. Given a strategy profile $s \in S$, for all $i \in N$ and $a_i \in A_i$ we define

$$\varphi_{i,a_i}(s) = \max\{0, u_i(a_i, s_{-i}) - u_i(s)\}.$$

We then define the function $f : S \to S$ by $f(s) = s'$, where

$$s'_i(a_i) = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{\sum_{b_i \in A_i} s_i(b_i) + \varphi_{i,b_i}(s)} = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}. \tag{5}$$

NASH EQUILIBRIUM

Theorem. [DGP'06, CDT'06]

Computing a Nash equilibrium in a 2-player game is PPAD-complete.

"Computing a Nash equilibrium
is as hard as
computing a general Brouwer fixed point."

COMPLEXITY OF CONTRACTION?

Contraction $(\varepsilon, \gamma = 1/2^{poly(n)})$

$\uparrow$

Simple Stochastic Game

$\uparrow$

Mean Payoff

$\uparrow$

Parity Game

QUERY MODEL

* We have a query access to the function f.

* Find an ε-approx. fixed point by as few queries as possible.

$|f(x)-x|_\infty \leq \varepsilon$

Efficient. $\mathrm{poly}(k, \log(1/\varepsilon), \log(1/\gamma))$.

POLY-QUERY ALGORITHM!

Our Main Result.

An $O(k^2 \cdot \log(1/\varepsilon))$ query algorithm for CONTRACTION(k, ε, γ).

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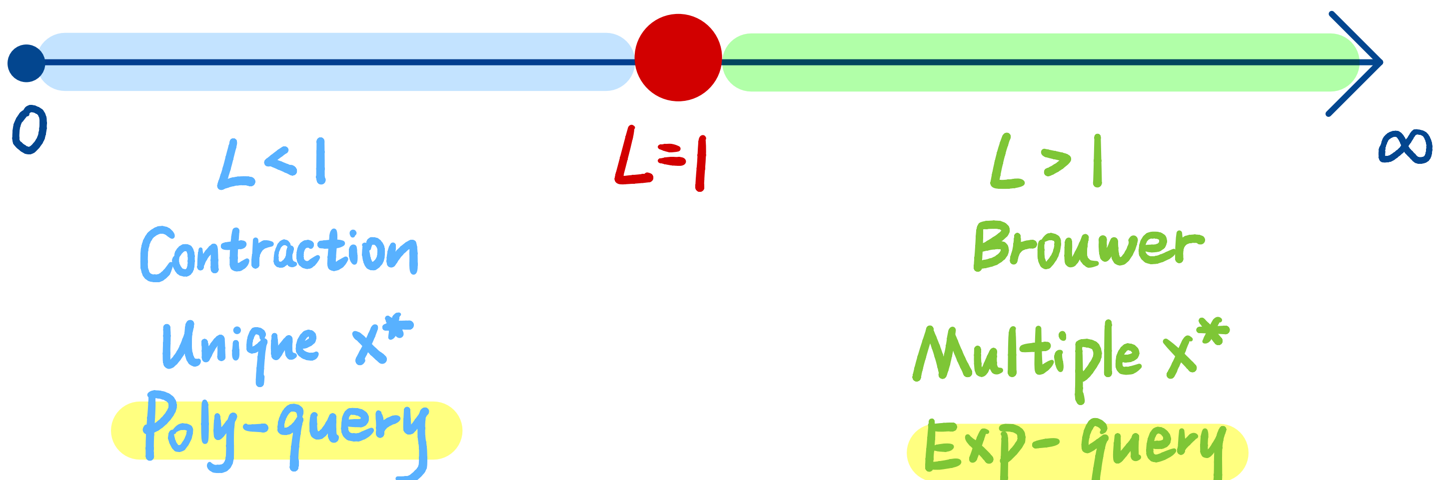

POLY-QUERY ALGORITHM!

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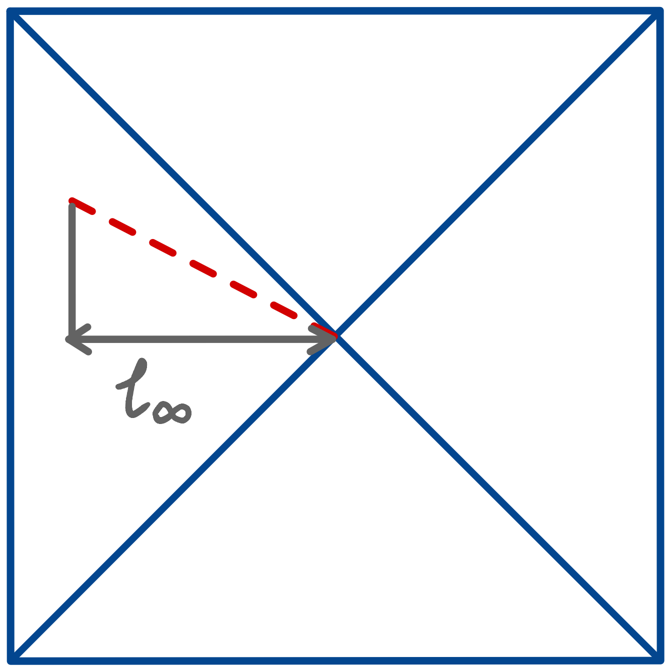
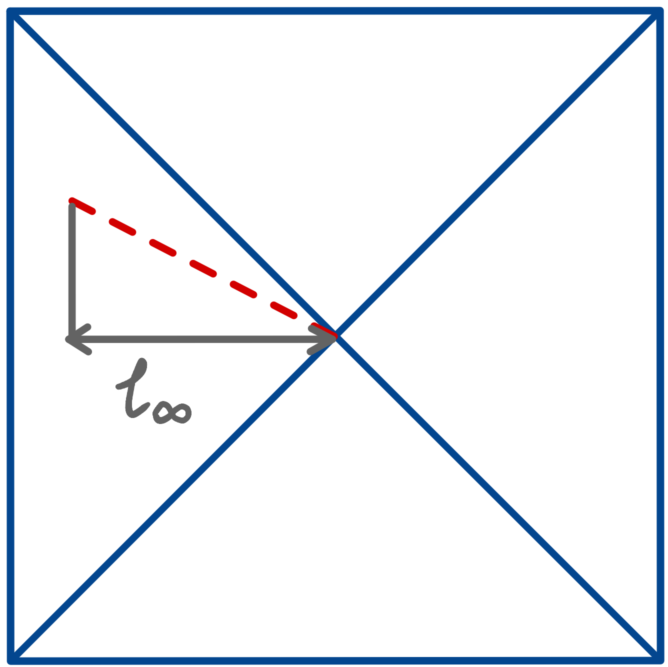
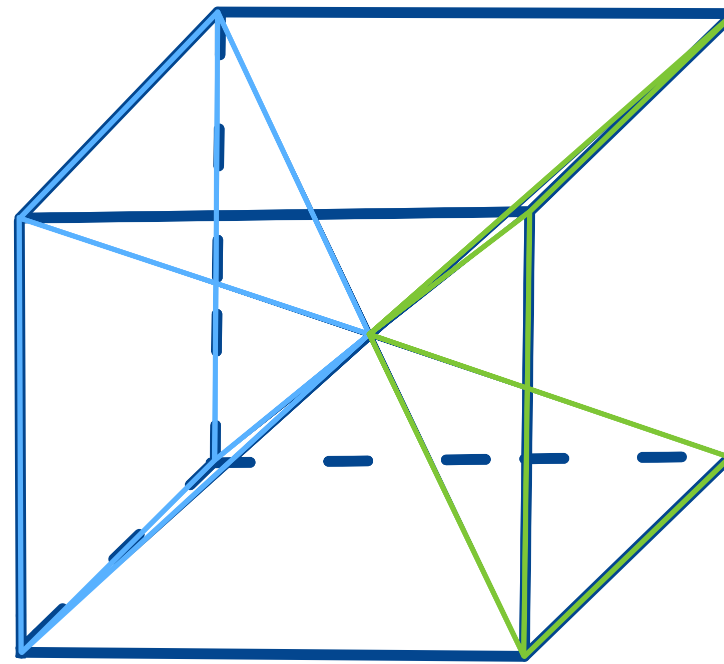
This makes contraction in a very intriguing complexity status!

TECHNIQUES



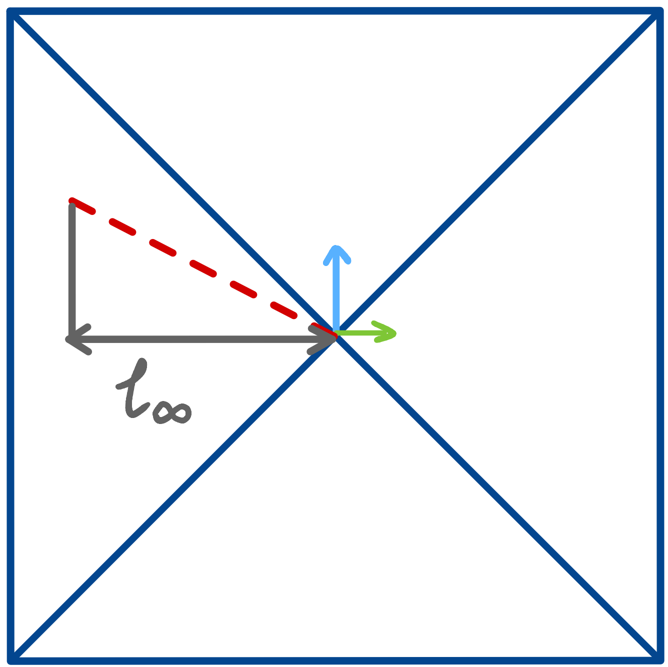
TECHNIQUES

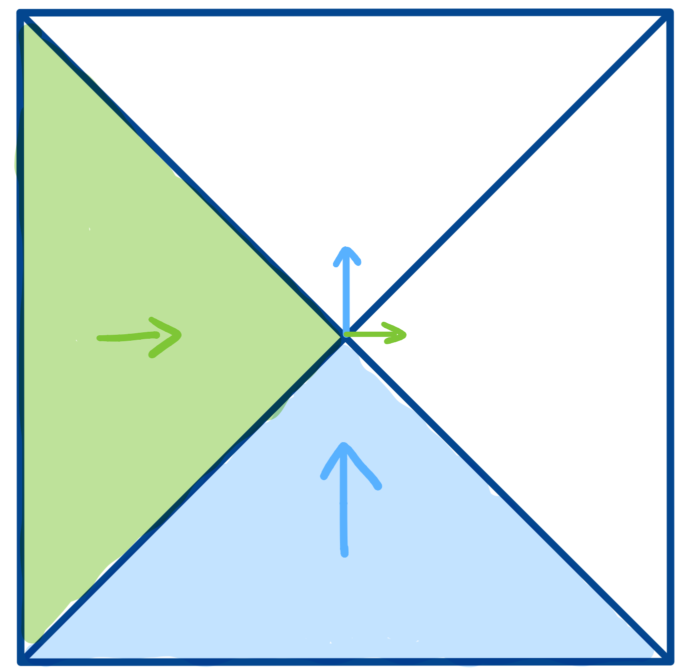




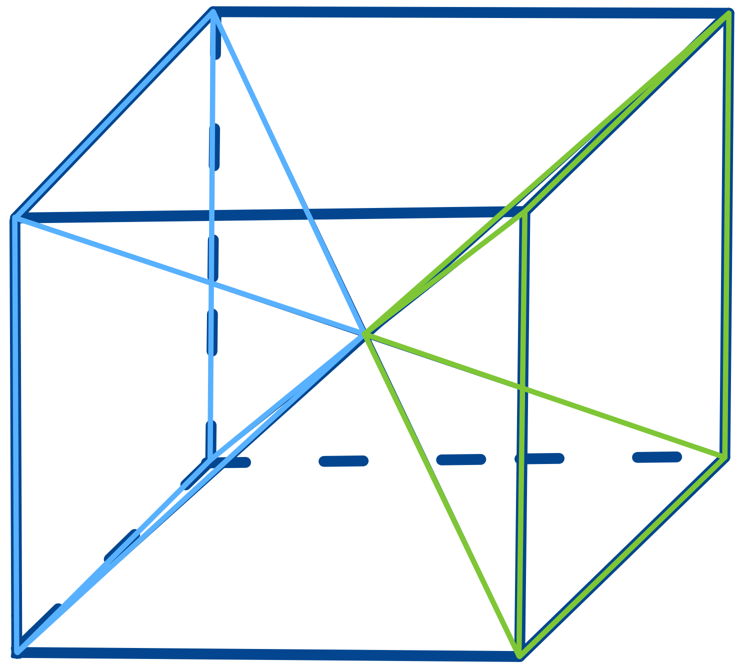
Pyramid

TECHNIQUES

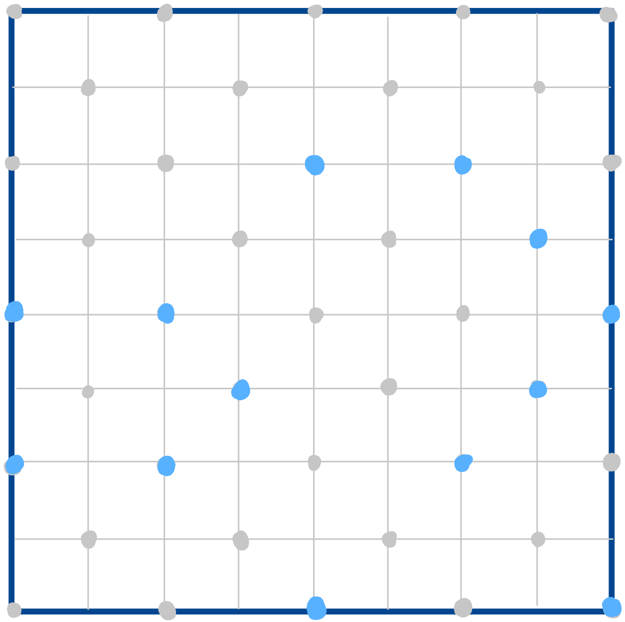




NON-CONVEX FOR 3-D



BALANCED POINT



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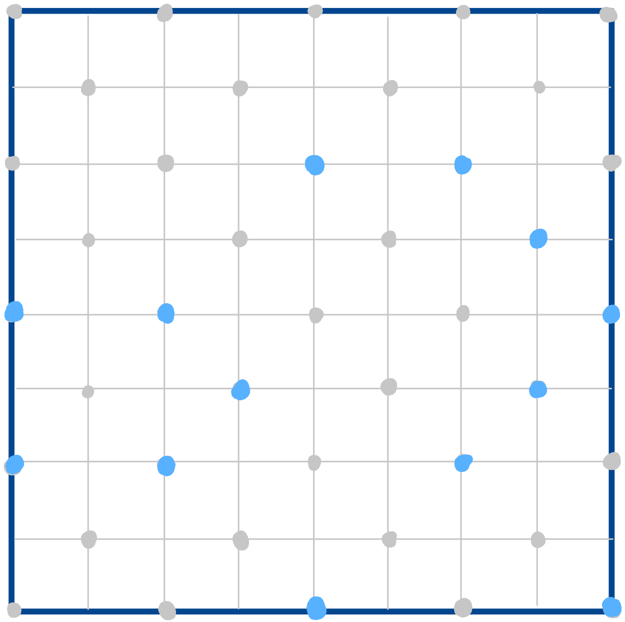
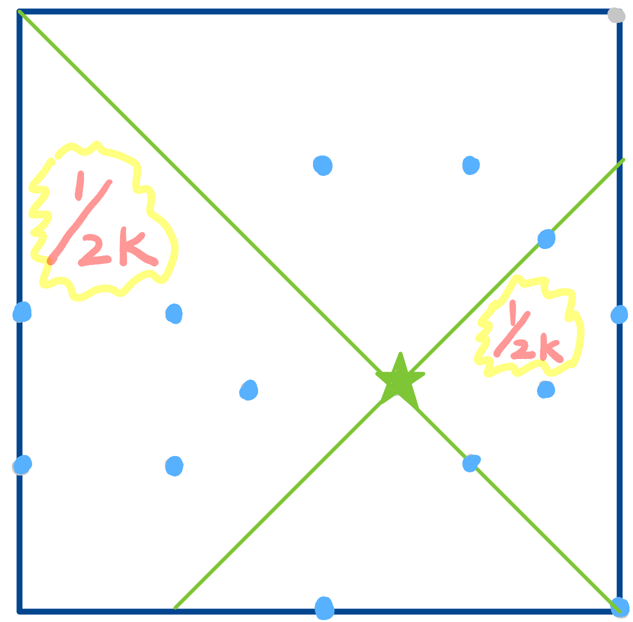
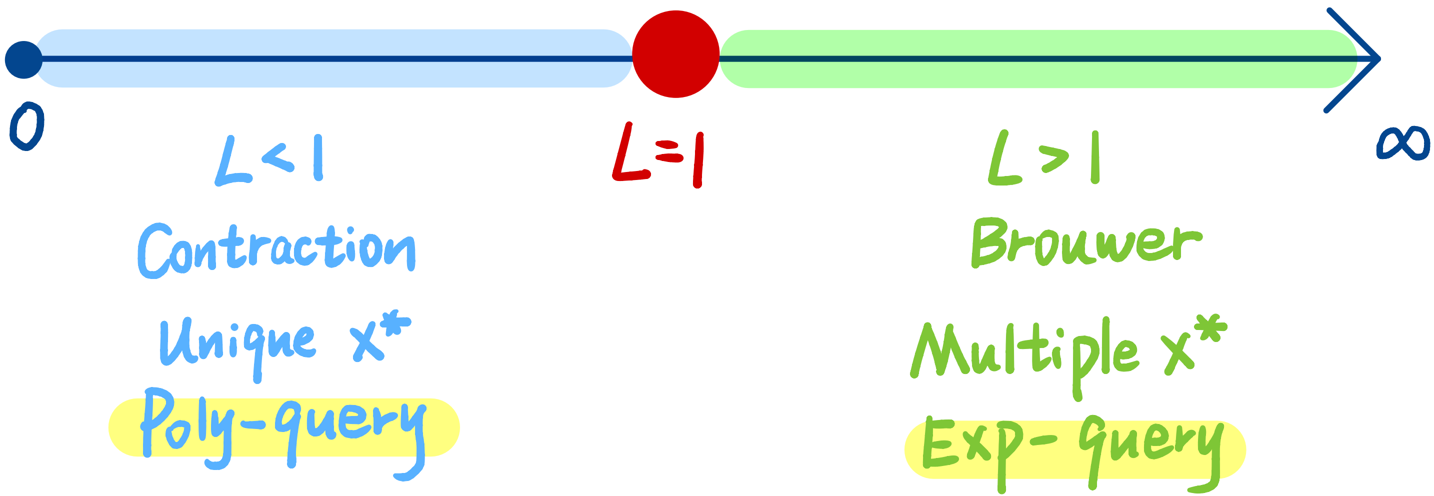
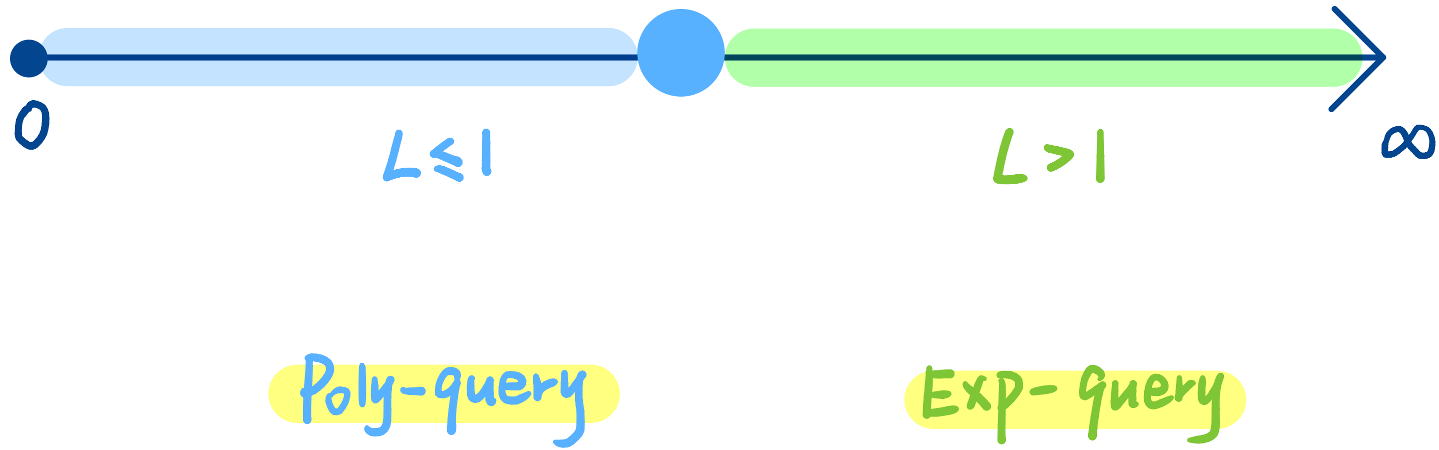




HOW ABOUT $L=1$?

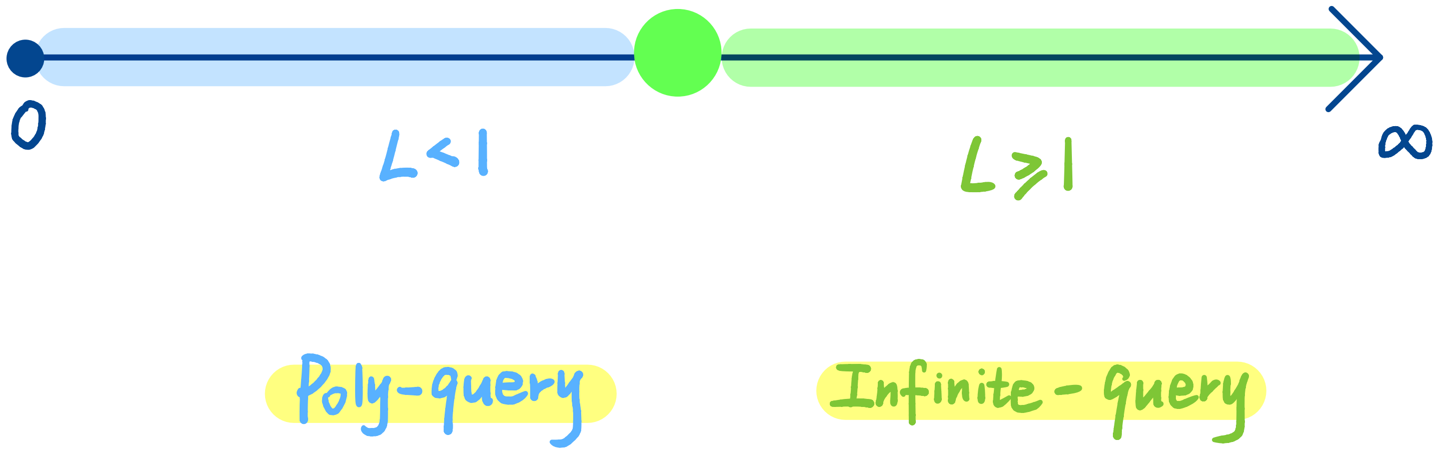

0
∞
$L<1$
Contraction
Unique x^*
Poly-query
$L=1$
$L>1$
Brouwer
Multiple x^*
Exp-query

WEAK APPROXIMATION



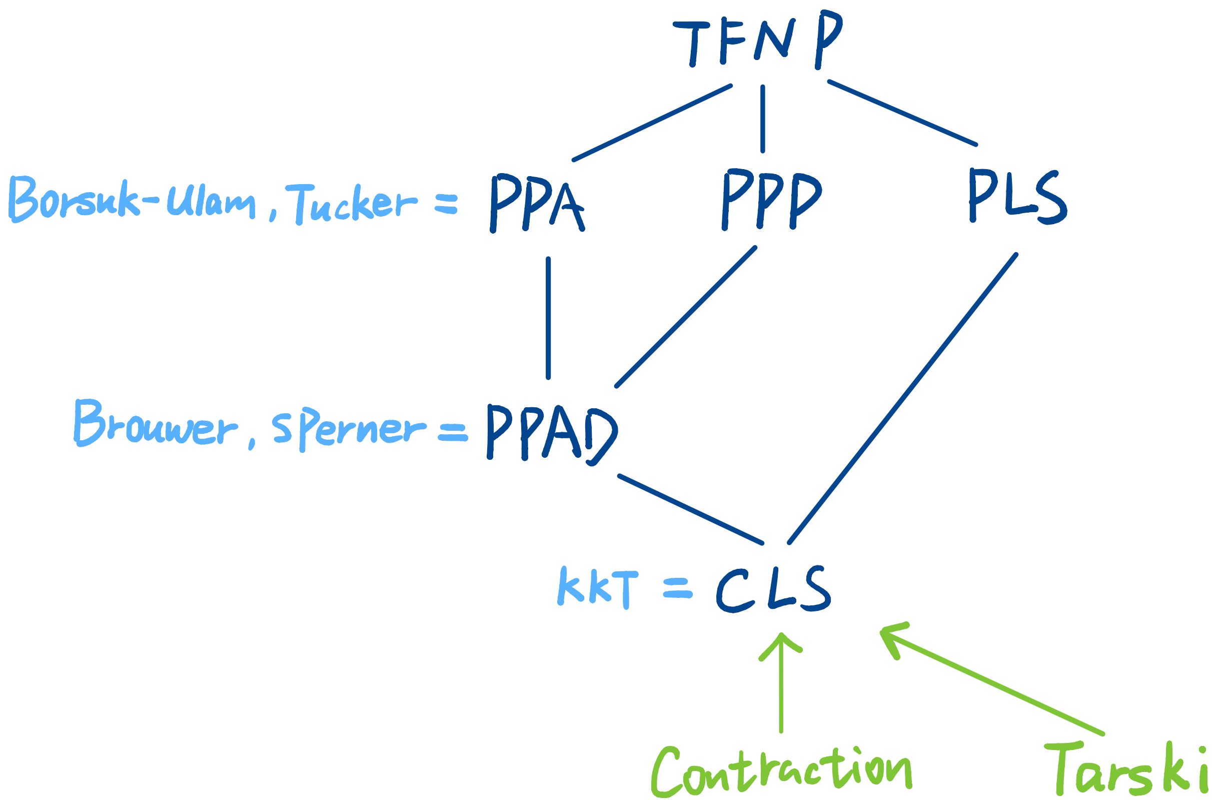
Weak approximation: $\|f(x) - x\|_\infty \leq \varepsilon$

STRONG APPROXIMATION



Strong approximation: $|x - x^*|_\infty \leq \varepsilon$

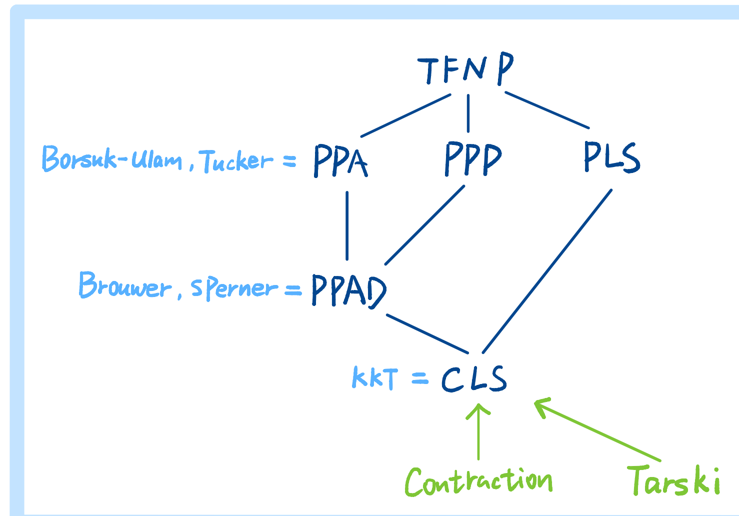
FIXED POINT COMPUTATION



INTRIGUING STATUS

* In CLS = PLS ∩ PPAD
* Not known query lower bound

Tarski
Contraction

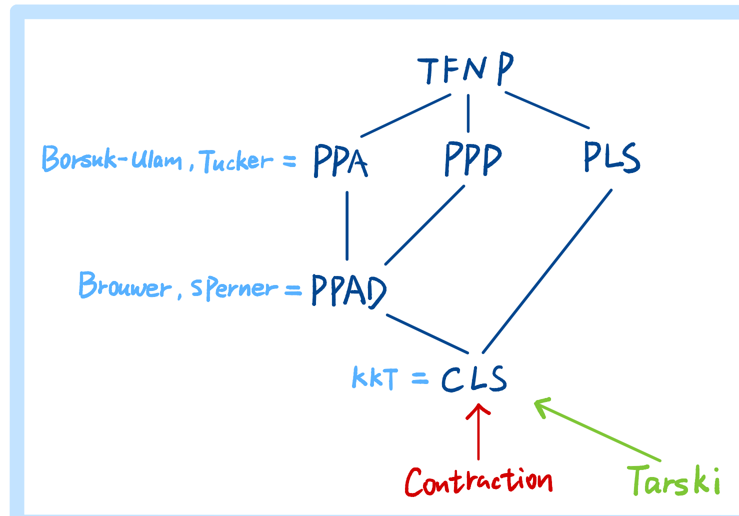


CONTRACTION: MORE INTRIGUING

* In CLS = PLS ∩ PPAD

* ~~Not known query lower bound~~

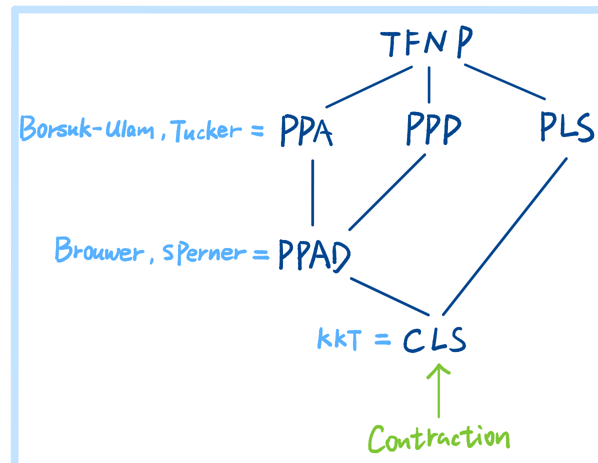
* Query lower bound is impossible!



INTERPRETATION

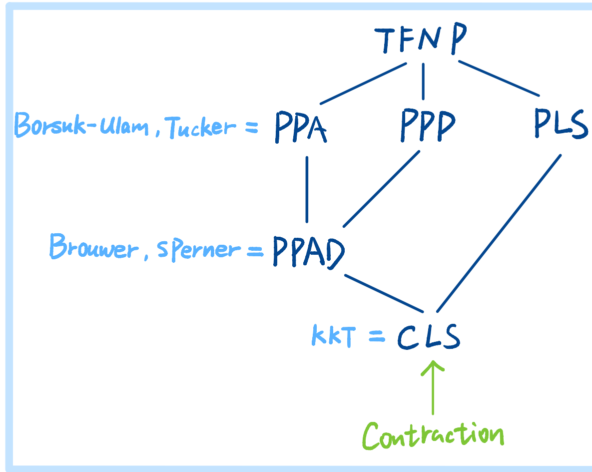
* All other fixed points that are complete for their corresponding classes have exponential query L.B.

* The story for contraction is completely different.



INTERPRETATION

① Hardness? Need to go beyond traditional wisdom about hardness in TFNP.

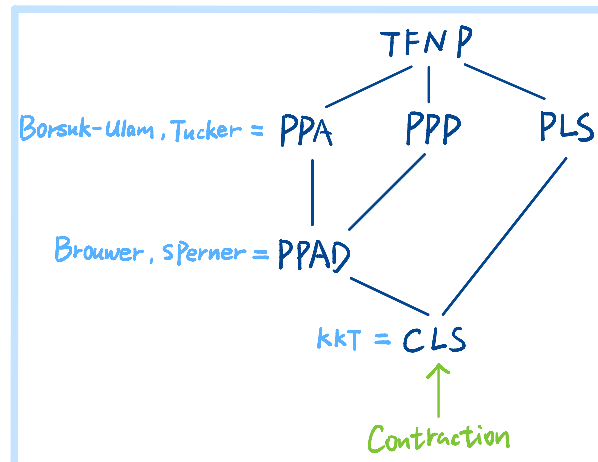


INTERPRETATION

① Hardness? Need to go beyond traditional wisdom about hardness in TFNP.

② We hope that it helps design time-efficient algs for contraction/SSGs.

Ultimately, poly-time algs.



OPEN PROBLEMS

* Time complexity for contraction.
* How about other p- norm ?

  The only known result is poly-query and poly-time algorithm for 2-norm. [STW'93, HKS'99]

THANKS !

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